SIC POVMs and Clifford groups in prime dimensions

Huangjun Zhu

Centre for Quantum Technologies, National University of Singapore, Singapore 117543, Singapore

NUS Graduate School for Integrative Sciences and Engineering, Singapore 117597, Singapore

E-mail: `zhuhuangjun@nus.edu.sg`

Abstract. We show that in prime dimensions not equal to three, each group covariant symmetric informationally complete positive operator valued measure (SIC POVM) is covariant with respect to a unique Heisenberg–Weyl (HW) group. Moreover, the symmetry group of the SIC POVM is a subgroup of the Clifford group. Hence, two SIC POVMs covariant with respect to the HW group are unitarily or antiunitarily equivalent if and only if they are on the same orbit of the extended Clifford group. In dimension three, each group covariant SIC POVM may be covariant with respect to three or nine HW groups, and the symmetry group of the SIC POVM is a subgroup of at least one of the Clifford groups of these HW groups respectively. There may exist two or three orbits of equivalent SIC POVMs for each group covariant SIC POVM, depending on the order of its symmetry group. We then establish a complete equivalence relation among group covariant SIC POVMs in dimension three, and classify inequivalent ones according to the geometric phases associated with fiducial vectors. Finally, we uncover additional SIC POVMs by regrouping of the fiducial vectors from different SIC POVMs which may or may not be on the same orbit of the extended Clifford group.

PACS numbers: 03.65.-w, 03.65.Wj, 02.10.De, 03.67.-a

Contents

1. Introduction

A positive operator valued measure (POVM) is the most general measurement in quantum theory. It consists of a set of outcomes represented mathematically as a set of positive operators Π_js satisfying $\sum_j \Pi_j = I$, where I is the identity. Given an input quantum state ρ, the probability of obtaining the outcome Π_j is given by $p_j = \mathrm{tr}(\rho\Pi_j)$. An *informationally complete* POVM (IC POVM) is one which allows us to reconstruct any quantum state according to the set of probabilities p_js. Simple parameter counting shows that an IC POVM contains at least d^2 outcomes for a d-dimensional Hilbert space. An IC POVM with d^2 outcomes is called *minimal*.

A *symmetric informationally complete* POVM (SIC POVM) [1–4] is a special minimal IC POVM which consists of d^2 pure subnormalized projectors with equal pairwise fidelity. It is considered as a fiducial POVM due to its high symmetry and high tomographic efficiency [2, 3, 5–7]. The existence of SIC POVMs in any finite dimension was first conjectured by Zauner [1] about ten years ago, and has been confirmed numerically in dimensions up to 67 [2, 4]. Analytical solutions have been found in dimensions 2, 3 [8]; 4, 5 [1]; 6 [9]; 7 [3]; 8 [10, 11]; 9, . . ., 15 [4, 11–14]; 19 [3];

24, 35, 48 [4]. It is generally believed that SIC POVMs exist in any finite dimension, however, a rigorous mathematical proof is not known.

In addition to their application in quantum state tomography, SIC POVMs are also interesting for many other reasons. They are closely related to mutually unbiased bases (MUB) [15–17]. They are studied under the name of equiangular lines [18] in the mathematical community, and are well known as minimal 2-design in design theory [2]. The Lie algebraic significance of SIC POVMs was also discussed recently [19].

A *group covariant* SIC POVM is one which can be generated from a single vector—*fiducial vector*—under the action of a group consisting of unitary operations. Almost all known SIC POVMs are covariant with respect to the *Heisenberg–Weyl (HW) group* or generalized Pauli group. The *Clifford group* is the normalizer of the HW group which consists of unitary operations, and the *extended Clifford group* is the larger group which contains also antiunitary operations [3,20]. Obviously, a fiducial vector remains a fiducial vector when transformed by any element in the extended Clifford group. Fiducial vectors and SIC POVMs form disjoint orbits under the action of the (extended) Clifford group. SIC POVMs on the same orbit of the extended Clifford group are *unitarily or antiunitarily equivalent* in the sense that they can be transformed into each other with unitary or antiunitary operations.

Except in a few small dimensions, there are generally more than one orbits of SIC POVMs according to the numerical searches performed by Scott and Grassl [4]. Hence, a natural question arise: Are two SIC POVMs on two different orbits of the (extended) Clifford group equivalent? This question is closely related to the following open question: For an HW covariant SIC POVM, is its (extended) symmetry group a subgroup of the (extended) Clifford group? By *symmetry group* (*extended symmetry group*) of a SIC POVM, we mean the set of all unitary (unitary or antiunitary) operations which leave the SIC POVM invariant. Although an affirmative answer to the later question is tacitly assumed by many researchers in the community, a rigorous proof is yet unavailable in the literature. In this paper we settle to answer all these questions for prime dimensions.

Dimension three is the only known case where there exist continuous orbits of SIC POVMs. Despite the low dimension, and the fact that fiducial vectors have been known for a long time [1–3], a complete picture of SIC POVMs in dimension three has yet to be uncovered. We shall show that there are some additional peculiarities about SIC POVMs in dimension three in contrast with the ones in other prime dimensions. Moreover, we shall provide new insights about SIC POVMs in dimension three by establishing a complete equivalence relation among all group covariant SIC POVMs and classifying all inequivalent ones according to the *geometric phases* [21,22] or Bargmann invariants [23] associated with fiducial vectors. In addition, we uncover additional SIC POVMs through regrouping of the fiducial vectors from different SIC POVMs which may or may not be on the same orbit of the extended Clifford group.

The paper is organized as follows. In section 2, we recall the basic properties of the SIC POVMs and Clifford groups in prime dimensions. In section 3, we prove that,

in any prime dimension, each group covariant SIC POVM is covariant with respect to the HW group, and the symmetry group of the SIC POVM is a subgroup of certain Clifford group. The implications of these results on the equivalence relation among group covariant SIC POVMs are also discussed in detail. In section 4, we establish a complete equivalence relation among all group covariant SIC POVMs in dimension three, and classify inequivalent ones. We also uncover additional SIC POVMs by regrouping of the fiducial vectors. We conclude with a summary.

2. Preliminary about SIC POVMs and Clifford groups in prime dimensions

2.1. SIC POVMs and Heisenberg–Weyl group

In a d-dimensional Hilbert space, a SIC POVM [1–3] consists of d^2 outcomes that are subnormalized projectors onto pure states $\Pi_j = \frac{1}{d}|\psi_j\rangle\langle\psi_j|$ for $j = 1, \ldots, d^2$, such that

$$|\langle\psi_j|\psi_k\rangle|^2 = \frac{1 + d\delta_{jk}}{d+1}. \tag{1}$$

The symmetry group G_{sym} of a SIC POVM consists of all unimodular unitary operators (unimodular operators or matrices are those with determinant 1) that leave the SIC POVM invariant, that is, permute the set of vectors $|\psi_j\rangle$s up to some phase factors. Since operators which differ only by overall phase factors implement essentially the same transformation, they can be identified into equivalence classes. Let G be any unitary group, and U any element in G. Throughout this paper, $\bar{U}$ is used to denote the equivalence class of U, and the expression $U' \in \bar{U}$ means that U' is in the equivalence class of U. With the product rule $\bar{U}\bar{U}' = \overline{UU'}$, the set of $\bar{U}$s form the *collineation group* $\bar{G}$ of G [24]. Let $\Phi(G)$ be the subgroup of G consisting of elements which are proportional to the identity; then $\bar{G}$ is the quotient group of G with respect to $\Phi(G)$, that is, $\bar{G} = G/\Phi(G)$. There exist infinitely many different unitary groups G with the same collineation group $\bar{G}$. However, if G is unimodular, then $|G| \leq d|\bar{G}|$, where $|G|$ ($|\bar{G}|$) denotes the order of G ($|\bar{G}|$). Moreover, there is a unique unimodular unitary group G' that satisfies $\bar{G}' = \bar{G}$ and $|G'| = d|\bar{G}|$ [24]. When there is no confusion, G and $\bar{G}$ will be referred to with the same name. For example $\bar{G}_{\text{sym}}$ is also called the symmetry group of the SIC POVM, and it is generally more convenient to work with $\bar{G}_{\text{sym}}$ rather than G_{sym}. We shall some times work with unimodular unitary groups and some times with collineation groups, depending on which one is more convenient. To simplify the notation, we shall often denote an element in the collineation group with a representative which need not be unimodular.

The group $\bar{G}_{\text{sym}}$ can also be defined in an alternative way. Let G'_{sym} be the group consisting of all unitary operations that leave the SIC POVM invariant, then $\bar{G}_{\text{sym}} = \bar{G}'_{\text{sym}} = G'_{\text{sym}}/\Phi(G'_{\text{sym}})$. The group G'_{sym} is also called the symmetry group of the SIC POVM. The advantage of the second definition is that it can be extended to cover antiunitary operations unambiguously. The extended symmetry group EG'_{sym} of a SIC POVM consists of all unitary or antiunitary operations that leave the SIC POVM

invariant. The quotient group $\overline{EG}'_{\rm sym} = EG'_{\rm sym}/\Phi(EG'_{\rm sym})$ is also called the extended symmetry group. To impose the unimodular constraint on the symmetry group is mainly to ensure that the group $G_{\rm sym}$ be finite, which is crucial in later discussions. In the rest of the paper, concerning the symmetry group of a SIC POVM, we only consider the collineation group and unimodular unitary group; concerning the extended symmetry group, we only consider the collineation group, and we write $\overline{EG}_{\rm sym}$ in place of $\overline{EG}'_{\rm sym}$ to simplify the notation.

Since a SIC POVM is informationally complete, any unitary operator that stabilizes all fiducial vectors must be proportional to the identity. Hence, the action of $\bar{G}_{\rm sym}$ on the set of vectors in the SIC POVM is faithful, which implies that $\bar{G}_{\rm sym}$ is isomorphic to a subgroup of the full symmetry group of d^2 letters, and is thus a finite group. As a result, $G_{\rm sym}$ is also a finite group due to the relation $|G| \leq d|\bar{G}|$.

Under the action of $\bar{G}_{\rm sym}$, the vectors in the SIC POVM form disjoint orbits. The *stability group* or stabilizer of a vector in the SIC POVM is the group consisting of all operations that leave the vector invariant. A SIC POVM is group covariant if the vectors of the SIC POVM form a single orbit under the action of $\bar{G}_{\rm sym}$. In that case, the SIC POVM is covariant with respect to $G_{\rm sym}$ or $\bar{G}_{\rm sym}$, and each vector in the SIC POVM is a fiducial vector. More generally, if the fiducial vectors of a group covariant SIC POVM are on the same orbit of a subgroup $\bar{G}$ of $\bar{G}_{\rm sym}$, then the SIC POVM is covariant with respect to G or $\bar{G}$. For a group covariant SIC POVM, the stability group of each fiducial vector is conjugated to each other, and $|\bar{G}_{\rm sym}|/|\bar{K}| = d^2$, where $\bar{K}$ is the stability group of any fiducial vector.

Two SIC POVMs are unitarily or antiunitarily equivalent if there exists a unitary or antiunitary operator that maps the vectors of one SIC POVM to that of the other one up to a permutation of the vectors in addition to some phase factors.

Most known SIC POVMs are covariant with respect to the Heisenberg–Weyl (HW) group or generalized Pauli group D [1–3], which is generated by the two operators X, Z defined below,

$$
\begin{aligned}
Z|e_r\rangle &= \omega^r |e_r\rangle, \\
X|e_r\rangle &= \begin{cases} |e_{r+1}\rangle & r = 0, 1, \ldots, d-2, \\ |e_0\rangle & r = d-1, \end{cases} \\
D_{k_1,k_2} &= \tau^{k_1 k_2} X^{k_1} Z^{k_2},
\end{aligned}
\tag{2}
$$

where $\omega = \mathrm{e}^{2\pi \mathrm{i}/d}$, $\tau = -\mathrm{e}^{\pi \mathrm{i}/d}$, $k_1, k_2 \in Z_d$, and Z_d is the additive group of integer modulo d. The group D consists of d^3 elements: $\omega^{k_3} D_{k_1,k_2}$ for $k_1, k_2, k_3 \in Z_d$, while $\bar{D}$ consists of d^2 elements: D_{k_1,k_2} for $k_1, k_2 \in Z_d$ (here we denote the elements in the collineation group with the representatives in the linear group). In addition, $\bar{D}$ is abelian, while D is not. Due to the commutation relation $XZX^{-1}Z^{-1} = \omega^{-1} I$, any unitary group with $\bar{D}$ as its collineation group must contain the subgroup generated by ωI. Moreover, there exists a unique unimodular unitary group with $\bar{D}$ as its collineation group, this unique unimodular unitary group is also referred to as the HW group. When d is odd, which is the most relevant case in this paper, the group D defined in (2) is already unimodular.

When d is even, it can be made unimodular by some phase factors. For example, in dimension two, the unimodular form of the HW group consists of the following eight elements $\pm I, \pm iX, \pm iZ, \pm XZ$.

A fiducial vector $|\psi\rangle$ of the HW group satisfies the following equation [1–3],

$$|\langle\psi|D_{\mathbf{k}}|\psi\rangle| = \frac{1}{\sqrt{d+1}} \quad \text{for} \quad \mathbf{k} \neq 0, \tag{3}$$

where $\mathbf{k} = (k_1, k_2)^T$.

The Clifford group $\mathrm{C}(d)$ is the normalizer of the HW group which consists of unimodular unitary operators; its collineation group $\bar{\mathrm{C}}(d)$ is also called the Clifford group. As in the case of HW group, $|\mathrm{C}(d)| = d|\bar{\mathrm{C}}(d)|$. The extended Clifford group $\overline{\mathrm{EC}}(d)$ is the larger group which contains also antiunitary operators, and it is generated by $\bar{\mathrm{C}}(d)$ and the complex conjugation operator $\hat{J}$ [3, 20]. By definition, for any element U in the extended Clifford group, $U|\psi\rangle$ is a fiducial vector whenever $|\psi\rangle$ is [3]. Fiducial vectors and SIC POVMs form disjoint orbits under the action of the (extended) Clifford group. SIC POVMs on the same orbit of the extended Clifford group are unitarily or antiunitarily equivalent. For SIC POVMs on different orbits, there is no simple criterion so far for determining their equivalence relation. In this paper we shall solve this problem for prime dimensions.

In the rest of the paper, except when stated otherwise, we assume that the dimension of the Hilbert space is a prime p, and we are only concerned with group covariant SIC POVMs. In prime dimensions, for any unimodular unitary group G, either $|G| = |\bar{G}|$ or $|G| = p|\bar{G}|$ is satisfied [24]. Most unimodular unitary groups we shall consider contain the HW group as a subgroup and thus contain also the subgroup generated by ωI; hence $|G| = p|\bar{G}|$. As a consequence, there is a one-to-one correspondence between unimodular unitary groups and their collineation groups.

It turns out that, in any prime dimension, a group covariant SIC POVM must be covariant with respect to the HW group (see section 3.1). Thus the Clifford group plays a crucial role in classifying group covariant SIC POVMs. In the rest of this section we focus on the Clifford groups in *odd prime* dimensions (see also [3, 20]), in preparation for the discussion in the next section. Some results presented here may also be of independent interests. The Clifford group in dimension two will be discussed briefly in section 3.2. Before discussing the Clifford group, we need to take a detour reviewing the properties of the special linear group $\mathrm{SL}(2, p)$ for odd prime p.

2.2. Special linear group $\mathrm{SL}(2, p)$

The special linear group $\mathrm{SL}(2, p)$ over the field Z_p consists of 2×2 matrices with entries from Z_p, and unit determinant mod p, that is elements of the form

$$F = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{4}$$

where $\alpha\delta - \beta\gamma = 1 \bmod p$. Likewise, the extended special linear group $\mathrm{ESL}(2, p)$ is the larger group which contains also 2×2 matrices with determinant $-1 \bmod p$. The orders

Table 1. Class representatives, and their numbers of conjugates in $\mathrm{SL}(2,p)$ for odd prime p from Humphreys [27], orders of these class representatives are also included for completeness. Here the class representatives $\mathbf{1}, z, c_1, c_2, a$ are defined in (6) (the class representatives c_1, c_2 are modified for convenience), b is an element of order $p+1$, $1 \le l \le \frac{p-3}{2}, 1 \le m \le \frac{p-1}{2}$ and $\gcd(l, p-1)$ denotes the greatest common divisor of l and $p-1$, similarly for $\gcd(m, p+1)$.

Representative	$\mathbf{1}$	z	a^l	b^m	c_1	c_2	zc_1	zc_2
Order	1	2	$\frac{p-1}{\gcd(l,p-1)}$	$\frac{p+1}{\gcd(m,p+1)}$	p	p	$2p$	$2p$
Number of conjugates	1	1	$p(p+1)$	$p(p-1)$	$\frac{1}{2}(p^2-1)$	$\frac{1}{2}(p^2-1)$	$\frac{1}{2}(p^2-1)$	$\frac{1}{2}(p^2-1)$

of $\mathrm{SL}(2,p)$ and $\mathrm{ESL}(2,p)$ are $p(p^2-1)$ and $2p(p^2-1)$ respectively. Since $\mathrm{ESL}(2,p)$ is a union of two cosets, that is $\mathrm{ESL}(2,p) = \mathrm{SL}(2,p) \cup J\mathrm{SL}(2,p)$, where

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5}$$

it is enough to focus on $\mathrm{SL}(2,p)$ in the following discussion.

The centre of the special linear group $\mathrm{SL}(2,p)$ is generated by $\mathrm{diag}(-1,-1)$, the unique order 2 element in the group. The quotient group of $\mathrm{SL}(2,p)$ with respect to its centre—the projective special linear group $\mathrm{PSL}(2,p)$—is a simple group (a group without nontrivial normal subgroups) for $p \ge 5$ [25, 26].

There are $p+4$ conjugacy classes in $\mathrm{SL}(2,p)$ [20, 27, 28]. Table 1 shows the class representatives, their orders and numbers of conjugates determined by Humphreys [27], where

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad z = -\mathbf{1}, \qquad c_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad c_2 = \begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix}, \qquad a = \begin{pmatrix} \nu & 0 \\ 0 & \nu \end{pmatrix} \tag{6}$$

and ν is a primitive element in Z_p. For the class representative c_2 or zc_2, ν can also be any element in Z_p that is not a quadratic residue, that is not a square of any element in Z_p. There is one class (class representative z, actually only one element) with elements of order 2, two classes (class representatives c_1, c_2) with elements of order p, and two classes (class representatives zc_1, zc_2) with elements of order $2p$. For each divisor k of $p-1$ which is not equal to 2, the number of classes with elements of order k is equal to $\frac{1}{2}\varphi(k)$. Here $\varphi(k)$ is the number of elements of order k in any cyclic group whose order is divisible by k. It is also known as the Euler function which denotes the number of positive integers which are less than k and coprime with k. Similarly, for each divisor k of $p+1$ which is not equal to 2, the number of classes with elements of order k is $\frac{1}{2}\varphi(k)$.

There are $p+1$ Sylow p-subgroups in $\mathrm{SL}(2,p)$, $Q_1, \ldots, Q_{p+1}$, and all of them are conjugated to each other according to Sylow's theorem [26]. The normalizer N_js of Q_js

are also conjugated to each other. Suppose Q_1 consists of the following elements:

$$\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \qquad \text{for} \quad \gamma \in Z_p, \tag{7}$$

then the normalizer N_1 of Q_1 consists of the following elements:

$$\begin{pmatrix} \alpha & 0 \\ \gamma & \alpha^{-1} \end{pmatrix} \qquad \text{for} \quad \gamma \in Z_p, \quad \alpha \in Z_p^*, \tag{8}$$

where Z_p^* is the multiplicative group consisting of nonzero elements in Z_p, which is also cyclic [26]. The order of N_1 is $p(p-1)$; it is cyclic if $p = 3$ and not cyclic if $p \geq 5$. In addition, each subgroup of N_1 whose order is equal to $2p$ or is not a multiple of p is cyclic.

$\mathrm{SL}(2,p) \ltimes (Z_p)^2$ is the semidirect product group of $\mathrm{SL}(2,p)$ and $(Z_p)^2$ equipped with the following product rule:

$$(F_1, \chi_1) \circ (F_2, \chi_2) = (F_1 F_2, \chi_1 + F_1 \chi_2), \tag{9}$$

where $F_1, F_2 \in \mathrm{SL}(2,p)$, and $\chi_1, \chi_2 \in (Z_p)^2$. Similarly, $\mathrm{ESL}(2,p) \ltimes (Z_p)^2$ is defined with the same product rule. The orders of $\mathrm{SL}(2,p) \ltimes (Z_p)^2$ and $\mathrm{ESL}(2,p) \ltimes (Z_p)^2$ are $p^3(p-1)$ and $2p^3(p-1)$ respectively [3]. In the following discussion, we focus on $\mathrm{SL}(2,p) \ltimes (Z_p)^2$.

The conjugacy classes of $\mathrm{SL}(2,p) \ltimes (Z_p)^2$ can be determined based on the conjugacy classes of $\mathrm{SL}(2,p)$, see also [20, 28]. Since elements of the form $(\mathbf{1}, \chi)$ for $\chi \neq \mathbf{0}$ form a single conjugacy class, it remains to consider (F, χ) with $F \neq \mathbf{1}$. Due to the product rule in (9), it suffices to deal with the case where F is a class representative of $\mathrm{SL}(2,p)$ listed in table 1. Consider the following equality:

$$(1, \chi_1) \circ (F, \chi) \circ (1, \chi_1)^{-1} = (F, (1 - F)\chi_1 + \chi); \tag{10}$$

if $F \neq c_1, c_2$, then $1 - F$ is nonsingular, and we can choose $\chi_1 = -(1-F)^{-1}\chi$ to eliminate the term $(1 - F)\chi_1 + \chi$; hence (F, χ) is conjugated to $(F, \mathbf{0})$. If $F = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$, according to a similar argument, $\left(F, \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}\right)$ is conjugated to $\left(F, \begin{pmatrix} k_1 \\ 0 \end{pmatrix}\right)$; in addition, $\left(F, \begin{pmatrix} k_1 \\ 0 \end{pmatrix}\right)$ is conjugated to $\left(F, \begin{pmatrix} k_1' \\ 0 \end{pmatrix}\right)$ if and only if $k_1' = k_1$ or $k_1' = -k_1$.

The class representatives, their orders, and numbers of conjugates in $\mathrm{SL}(2,p) \ltimes (Z_p)^2$ are shown in table 2. There are $2p+4$ conjugacy classes, $p+2$ of which consist of elements of order p. The number of classes with elements of any other order is the same as that in $\mathrm{SL}(2,p)$. That is, one class of order 2, two classes of order $2p$, and $\frac{1}{2}\varphi(k)$ classes of order k for each divisor k of $p-1$ or $p+1$ which is not equal to 2, where $\varphi(k)$ is the Euler function. Note that $\varphi(k)$ is equal to 0, 1, 2, 2, 4, 2 respectively for $k = 1, \ldots, 6$, and $\varphi(k) > 2$ for any other positive integer k. It follows that, if $p > 3$, all order 3 elements are conjugated to each other, so are all order 4 elements and order 6 elements, and there are more than one classes if the order is a divisor of $p-1$ or $p+1$ which is equal to 5 or larger than 6.

Table 2. Class representatives, their orders and numbers of conjugates in $\mathrm{SL}(2,p) \ltimes (Z_p)^2$ for odd prime p. Here $\mathbf{1}, z, c_1, c_2, a$ are defined in (6), b is an element of order $p+1$ in $\mathrm{SL}(2,p)$, $1 \le l \le \frac{p-3}{2}$, $1 \le m \le \frac{p-1}{2}$, $1 \le k_1, k_2 \le \frac{p-1}{2}$ and $\gcd(l, p-1)$ denotes the greatest common divisor of l and $p-1$, similarly for $\gcd(m, p+1)$.

Representative	$(\mathbf{1},\mathbf{0})$	$\left(\mathbf{1},\begin{pmatrix}1\\0\end{pmatrix}\right)$	$(z,\mathbf{0})$	$(a^l,\mathbf{0})$	$(b^m,\mathbf{0})$	$(c_1,\mathbf{0})$
Order	1	p	2	$\frac{p-1}{\gcd(l,p-1)}$	$\frac{p+1}{\gcd(m,p+1)}$	p
Number of conjugates	1	p^2-1	p^2	$p^3(p+1)$	$p^3(p-1)$	$\frac{1}{2}p(p^2-1)$

Representative	$\left(c_1,\begin{pmatrix}k_1\\0\end{pmatrix}\right)$	$(c_2,\mathbf{0})$	$\left(c_2,\begin{pmatrix}k_2\\0\end{pmatrix}\right)$	$(zc_1,\mathbf{0})$	$(zc_2,\mathbf{0})$
Order	p	p	p	$2p$	$2p$
Number of conjugates	$p(p^2-1)$	$\frac{1}{2}p(p^2-1)$	$p(p^2-1)$	$\frac{1}{2}p^2(p^2-1)$	$\frac{1}{2}p^2(p^2-1)$

2.3. Clifford group

An important step towards understanding the structures of the (extended) Clifford groups and SIC POVMs is the following isomorphism given by Appleby [3],

$$\begin{aligned} &f_E : \mathrm{ESL}(p) \ltimes (Z_p)^2 \to \overline{\mathrm{EC}}(p), \\ &\quad U D_{\mathbf{k}} U^\dagger = \omega^{\langle \chi, F\mathbf{k}\rangle} D_{F\mathbf{k}} \qquad \text{for} \quad U = f_E(F,\chi), \end{aligned} \tag{11}$$

where $\langle \mathbf{k}, \mathbf{q}\rangle = k_2 q_1 - k_1 q_2$. Here is the explicit correspondence if $\det(F) = 1$ (assuming F is given in (4)) and $\beta \neq 0$,

$$\begin{aligned} &(F,\chi) \to U = D_\chi V_F, \\ &V_F = \frac{1}{\sqrt{p}} \sum_{r,s=0}^{p-1} \tau^{\beta^{-1}(\alpha s^2 - 2rs + \delta r^2)} |e_r\rangle\langle e_s|. \end{aligned} \tag{12}$$

If $\beta = 0$, then $\alpha, \delta \neq 0$, $\alpha\delta = 1$, and F can be written as the product of the following two matrices:

$$F_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad F_2 = \begin{pmatrix} \gamma & \delta \\ -\alpha & 0 \end{pmatrix}, \tag{13}$$

such that V_{F_1} and V_{F_2} can be computed according to (12). Hence we have $(F,\chi) \to D_\chi V_F = D_\chi V_{F_1} V_{F_2}$, where

$$\begin{aligned} &V_{F_1} = \frac{1}{\sqrt{p}} \sum_{r,s=0}^{p-1} \tau^{2rs} |e_r\rangle\langle e_s|, \\ &V_{F_2} = \frac{1}{\sqrt{p}} \sum_{r,s=0}^{p-1} \tau^{\delta^{-1}(\gamma s^2 - 2rs)} |e_r\rangle\langle e_s|, \end{aligned}$$

$$V_F = V_{F_1}V_{F_2} = \sum_{s=0}^{p-1} \tau^{\alpha\gamma s^2} |e_{\alpha s}\rangle\langle e_s|. \tag{14}$$

If $\det(F) = -1$, then $\det(JF) = 1$, and $(JF, \chi) \in \mathrm{SL}(2,p) \ltimes (Z_p)^2$. Hence the isomorphism images of elements in $\mathrm{ESL}(2,p) \ltimes (Z_p)^2$ can be determined once the images of elements in $\mathrm{SL}(2,p) \ltimes (Z_p)^2$ and that of $(J, \mathbf{0})$ are determined respectively. The isomorphism image of $(J, \mathbf{0})$ is the complex conjugation operator $\hat{J}$ [3].

Following Appleby, $[F, \chi]$ is used to denote the isomorphism image of (F, χ) under the correspondence (11) throughout this paper. In the following discussion, we focus on the Clifford group, except when otherwise stated.

Since the Clifford group and $\mathrm{SL}(2,p) \ltimes (Z_p)^2$ are isomorphic, they have the same class structure. There are also $2p + 4$ classes in the Clifford group, and the class representatives can be chosen as the isomorphism images of that of $\mathrm{SL}(2,p) \ltimes (Z_p)^2$ listed in table 2. There are $p+2$ classes with elements of order p, two classes with elements of order $2p$, one class with elements of order 2, and $\frac{1}{2}\varphi(k)$ classes with elements of order k for each divisor of $p-1$ or $p+1$ which is not equal to 2. In addition, if $p > 3$, all order 3 elements are conjugated to each other, recovering the result obtained by Flammia [29], so are all order 4 elements and order 6 elements. By contrast, there are more than one classes if the order is a divisor of $p-1$ or $p+1$ which is equal to 5 or larger than 6.

The spectrum, and in particular the dimension of each eigenspace of a Clifford unitary plays an important role in proving our main results in the next section. Here we give a brief account of the spectrum of the elements in each conjugacy class, see [20] for additional information.

The spectrum of each element in the class $\left[1, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$ is the same as that of Z, and is thus nondegenerate. For each element in the class $[z, \mathbf{0}]$, according to (14), there are two distinct eigenvalues ± 1 (all eigenvalues are defined up to an overall phase factor) with multiplicity $\frac{p\pm1}{2}$ respectively. For each element in the class $[a, \mathbf{0}]$, all eigenvalues are $(p-1)$st roots of unity; the eigenvalue 1 is doubly degenerate, and all other eigenvalues are nondegenerate. The spectrum of each element in the class $[a^l, \mathbf{0}]$ for $1 \le l \le \frac{p-3}{2}$ is simply the corresponding power of that of each element in the class $[a, \mathbf{0}]$.

The spectrum of each element in the class $[b, \mathbf{0}]$ can be determined in virtue of representation theory. Note that each element $F = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in the class b of $\mathrm{SL}(2,p)$ satisfies the two inequalities, $\beta \neq 0$ and $\alpha + \delta \neq 2$. According to (12),

$$|\mathrm{tr}(V_F)|^2 = \frac{1}{p}\left|\sum_{s=0}^{p-1} \tau^{\beta^{-1}(\alpha+\delta-2)s^2}\right|^2 = 1, \tag{15}$$

similarly, $|\mathrm{tr}[(V_F)^n]|^2 = |\mathrm{tr}(V_{F^n})|^2 = 1$ for $n = 1, \ldots, p$. If we take $(V_F)^n$ for $n = 0, \ldots, p$ as a representation of a cyclic group of order $p+1$, then the sum of squared multiplicities of all irreducible components in this representation is given by [30]

$$\frac{1}{p+1}\sum_{n=0}^{p} |\mathrm{tr}[(V_F)^n]|^2 = \frac{p^2+p}{p+1} = p. \tag{16}$$

Since all irreducible representations of any cyclic group are one dimensional, the above representation is a direct sum of p one-dimensional irreducible representations. Equation (16) implies that all the p irreducible components are distinct, which in turn implies the nondegeneracy of the spectrum of V_F. Each eigenvalue of V_F is a $(p+1)$st root of unity, hence the spectrum of V_F contains all but one $(p+1)$st roots of unity. The spectrum of each element in the class $[b^m, \mathbf{0}]$ for $1 \leq m \leq \frac{p-1}{2}$ is the corresponding power of the spectrum of each element in the class $[b, \mathbf{0}]$.

Now consider each element in either of the class $[c_1, \mathbf{0}]$ or $[c_2, \mathbf{0}]$. According to (14), if $\beta = 0, \alpha = \delta = 1$, that is $F \in Q_1$ (Q_1 is a Sylow p-subgroup of $\mathrm{SL}(2,p)$, see (7) for its definition), then V_F is diagonal with diagonal entries $\tau^{\gamma s^2}$ for $s = 0, \dots, (p-1)$. The distinct eigenvalues of V_F are $\tau^{\gamma s^2}$ for $s = 0, \dots, \frac{p-1}{2}$, and all eigenvalues are doubly degenerate except the eigenvalue 1. Thus for each element in the class $[c_1, \mathbf{0}]$, the distinct eigenvalues are τ^{s^2} for $s = 0, \dots, \frac{p-1}{2}$, and for each element in the class $[c_2, \mathbf{0}]$, they are $\tau^{\nu s^2}$ for $s = 0, \dots, \frac{p-1}{2}$, where ν is any element in Z_p^* that is not a quadratic residue. For each element in either of the two classes, all eigenvalues are doubly degenerate except the eigenvalue 1.

For each element in either of the class $\left[c_1, \begin{pmatrix} k_1 \\ 0 \end{pmatrix}\right]$ or $\left[c_2, \begin{pmatrix} k_2 \\ 0 \end{pmatrix}\right]$, direct inspection shows that its spectrum is the same as that of Z.

For each element in either the class $[zc_1, \mathbf{0}]$ or $[zc_2, \mathbf{0}]$, according to (14), all eigenvalues are distinct, which are given by 1, $\pm\tau^{s^2}$, or 1, $\pm\tau^{\nu s^2}$, for $s = 0, \dots, \frac{p-1}{2}$. The nondegeneracy of eigenvalues can also be shown using similar arguments as applied to the elements in the class $[b, \mathbf{0}]$. Suppose $F = zc_1$ or $F = zc_2$, according to (14), $|\mathrm{tr}[(V_F)^{2k-1}]|^2 = 1$ for $k = 1, \dots, p$, and $|\mathrm{tr}[(V_F)^{2k}]|^2 = p$ for $k = 1, \dots, (p-1)$. If we take $(V_F)^n$ for $n = 0, \dots, (2p-1)$ as a representation of a cyclic group of order $2p$, then the sum of squared multiplicities of all irreducible components in this representation is given by

$$\frac{1}{2p}\sum_{n=0}^{2p-1}|\mathrm{tr}[(V_F)^n]|^2 = \frac{p^2+p+p(p-1)}{2p} = p. \tag{17}$$

Hence, each irreducible component occurs only once, which implies the nondegeneracy of the spectrum of V_F.

A unitary matrix is a *monomial matrix* or *in monomial form* if there is only one nonzero entry in each row and each column. A unitary group is a *monomial group* if there exists a unitary transformation that simultaneously turns all elements in the group into monomial form. If every element in a monomial group is already monomial, then the group is *in monomial form*. According to (14), V_F is monomial if $\beta = 0$, that is $F \in N_1$ (N_js are defined in section 2.2), and it is a permutation matrix if in addition $\gamma = 0$.

Let H be a subgroup of the Clifford group that contains the HW group D. The quotient group H/D can be identified with a subgroup of $\mathrm{SL}(2,p)$. If $H/D \subset N_j$ with $1 \leq j \leq p+1$, then H is monomial, and it is already in monomial form if in addition

$j = 1$. What is not so obvious is that if H is monomial, then H/D is a subgroup of N_j with $1 \leq j \leq p+1$. To see this, let U be a unitary operator that brings H into the monomial form H', and D' be the image of D under the same transformation. Since D' can be turned into D with a suitable monomial unitary transformation, without loss of generality, we can assume that $D' = D$ and U is a Clifford unitary. Hence, $H'/D \in N_1$, and $H \in N_j$ with $1 \leq j \leq p+1$. As a consequence, when H is monomial, H/D is cyclic if its order is equal to $2p$ or not a multiple of p, according to the discussion in section 2.2.

Zauner's conjecture states that HW fiducial vectors exist in any finite dimension, and every such fiducial vector is an eigenvector of a canonical order 3 unitary [1–4] (there are several different versions of Zauner's conjecture [3], a specific one has been chosen here). Interestingly, when $3|(p-2)$, Zauner's conjecture implies that the symmetry group G_{sym} of a SIC POVM cannot be monomial. To demonstrate this point, let G'_{sym} be the intersection of G_{sym} with the Clifford group. If each fiducial vector is stabilized by an order 3 element in the Clifford group, then 3 divides $|G'_{\mathrm{sym}}/D|$, which in turn divides $|N_1|$ if G'_{sym} is monomial, according to the previous discussions. This, however, contradicts the fact that $|N_1| = p(p-1)$ is not divisible by 3.

2.4. Heisenberg–Weyl groups in the Clifford group

There are many other subgroups in the Clifford group that are unitarily equivalent to the HW group defined in (2); these groups will also be called HW groups. The normalizer of each of these groups will be referred to as the Clifford group of that HW group. If necessary, we will refer to the HW group defined in (2) as the *standard HW group*, and its (extended) Clifford group as the *standard (extended) Clifford group*. In this section, we focus on these additional HW groups and Clifford groups, since they play an important role in understanding the structure of SIC POVMs, as we shall see in section 3 and section 4.

For prime dimensions, each HW groups is a p-group, and is thus contained in a Sylow p-subgroup [26] of the Clifford group. In correspondence to the $p+1$ Sylow p-subgroups Q_j in $\mathrm{SL}(2,p)$, there are $p+1$ Sylow p-subgroups $\bar{P}_j(P_j)$ for $j = 1, \ldots, p+1$ in the Clifford group $\bar{\mathrm{C}}(p)$ ($\mathrm{C}(p)$), such that $\bar{P}_j/\bar{D} = Q_j$ ($P_j/D = Q_j$). The intersection of these Sylow p-subgroups is exactly the standard HW group.

Since all Sylow p-subgroups in the Clifford group are conjugated to each other, it suffices to study any one of them, say the Sylow p-subgroup $\bar{P}_1$ (P_1), which is generated by the following three elements:

$$V = \left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right], \qquad X = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right], \qquad Z = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]. \quad (18)$$

The order of $\bar{P}_1$ (P_1) is p^3 (p^4), and the order of any element in $\bar{P}_1$ other than identity is p. The centre of $\bar{P}_1$ is the cyclic group $\langle Z \rangle$ generated by Z, while the centre of P_1 is the cyclic group $\langle \omega I \rangle$ generated by ωI. Since each subgroup of P_1 of order p^3 necessarily contains the subgroup $\langle \omega I \rangle$, there is a one-to-one correspondence between the subgroups

of P_1 of order p^3 and subgroups of $\bar{P}_1$ of order p^2. There are $p+1$ order p^2 subgroups in $\bar{P}_1$, $\langle Z, V^jX\rangle$ for $j=0,\ldots,p-1$ and $\langle Z, V\rangle$. The first p of them are unitarily equivalent, as we shall see shortly.

According to (14),

$$V=\mathrm{diag}(1,\tau,\tau^4,\ldots,\tau^{(p-1)^2}),\qquad \det(V)=\tau^{p(p-1)(2p-1)/6}. \tag{19}$$

If $p\geq 5$, $(p-1)(2p-1)$ is divisible by 6, and $\det(V)$ is equal to $\tau^{p(p-1)(2p-1)/6}=1$, recall that $\tau=-\mathrm{e}^{\pi\mathrm{i}/p}$. Define

$$\begin{aligned} U &= \mathrm{diag}(1,\mathrm{e}^{\mathrm{i}\phi_1},\ldots,\mathrm{e}^{\mathrm{i}\phi_{p-1}}),\\ \mathrm{e}^{\mathrm{i}\phi_1} &= \tau,\\ \mathrm{e}^{\mathrm{i}\phi_2} &= \tau^{1+2^2},\\ &\ldots\\ \mathrm{e}^{\mathrm{i}\phi_{p-1}} &= \tau^{1+2^2+\cdots+(p-1)^2}=1; \end{aligned} \tag{20}$$

then we have

$$U^jZU^{j\dagger}=Z,\qquad U^jXU^{j\dagger}=V^jX,\qquad \text{for}\quad j=0,\ldots,p-1. \tag{21}$$

If $p=3$, then $\det(V)=\tau^{p(p-1)(2p-1)/6}=\tau^5=\mathrm{e}^{2\pi\mathrm{i}/3}$. Define $V'=\mathrm{e}^{4\mathrm{i}\pi/9}V$ and

$$U=\mathrm{diag}(1,\mathrm{e}^{-2\mathrm{i}\pi/9},\mathrm{e}^{-4\mathrm{i}\pi/9}); \tag{22}$$

then we have $\det(V')=1$ and

$$U^jZU^{j\dagger}=Z,\qquad U^jXU^{j\dagger}=V'^jX,\qquad \text{for}\quad j=0,1,2. \tag{23}$$

In conclusion, all the p groups $\langle Z, V^jX\rangle$ for $j=0,\ldots,p-1$ are unitarily equivalent to the standard HW group, and they are permuted cyclically under the transformation U in (20) for $p\geq 5$ or in (22) for $p=3$. The group $\langle Z,V\rangle$ cannot be unitarily equivalent to the HW group because all elements in the group are diagonal. Hence, there are $p(p+1)+1$ order p^2 (p^3) subgroups in the Clifford group $\bar{\mathrm{C}}(p)$ ($\mathrm{C}(p)$), out of which p^2 subgroups are unitarily equivalent to the HW group, recall that the standard HW group is the intersection of the $p+1$ Sylow p-subgroups of the Clifford group.

The p^2-1 additional HW groups in the Clifford group form a single orbit if $p=3$ or $3|(p-2)$, and three orbits if $3|(p-1)$. To demonstrate this point, it is enough to show that the $p-1$ additional HW groups in the Sylow p-subgroup $\bar{P}_1$ form the corresponding number of orbits in the two cases respectively, because all Sylow p-subgroups are conjugated to each other. Suppose the two HW groups $\langle Z, V^lX\rangle$ and $\langle Z, V^jX\rangle$, where $1\leq l,j\leq p-1$, are connected by the Clifford unitary $\left[\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix},\begin{pmatrix}k_1\\ k_2\end{pmatrix}\right]$; then the Clifford unitary belongs to the normalizer of $\bar{P}_1$, and $\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}$ belongs to the normalizer N_1 of Q_1 (see (7) and (8) for the definitions of Q_1 and N_1 respectively), which implies that $\beta=0$, and $\delta=\alpha^{-1}$. Recall that $V^lX=\left[\begin{pmatrix}1&0\\ l&1\end{pmatrix},\begin{pmatrix}1\\ l\end{pmatrix}\right]$, we have

$$V_1=\left[\begin{pmatrix}\alpha&0\\ \gamma&\alpha^{-1}\end{pmatrix},\begin{pmatrix}k_1\\ k_2\end{pmatrix}\right]V^lX\left[\begin{pmatrix}\alpha&0\\ \gamma&\alpha^{-1}\end{pmatrix},\begin{pmatrix}k_1\\ k_2\end{pmatrix}\right]^{-1}=\left[\begin{pmatrix}1&0\\ \alpha^{-2}l&1\end{pmatrix},\begin{pmatrix}\alpha\\ l'\end{pmatrix}\right], \tag{24}$$

where $l' \in Z_p$, whose specific value is not important here. Hence, $V_1 \in \langle Z, V^j X\rangle$ if and only if $\alpha^{-3} l = j$. If $p = 3$ or $3|(p-2)$, then α^{-3} may take any value in Z_p^*, and there exists α satisfying $\alpha^{-3} l = j$ for any pair $l, j \in Z_p^*$. As a result, the $(p-1)$ HW groups $\langle Z, V^l X\rangle$ for $l = 1, \dots, p-1$ are on the same orbit. If $3|(p-1)$, then α^{-3} may only take one third possible values in Z_p^*. Hence, the $(p-1)$ HW groups form three orbits of equal length $\frac{p-1}{3}$.

Suppose $\bar{D}'$ is any HW group in $\bar{P}_1$ other than the standard one, and $\bar{\mathrm{C}}'(p)$ its Clifford group. Our analysis in the last paragraph also shows that the normalizer of $\bar{D}'$ within $\bar{\mathrm{C}}(p)$ consists of the following elements:

$$\left[\begin{pmatrix} \alpha & 0 \\ \gamma & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}\right] \qquad \text{with} \quad \alpha, \gamma, k_1, k_2 \in Z_p, \quad \alpha^3 = 1. \tag{25}$$

If $p = 3$ or $3|(p-2)$, this group is exactly $\bar{P}_1$, that is, $\bar{\mathrm{C}}'(p) \cap \bar{\mathrm{C}}(p) = \bar{P}_1$; if $3|(p-1)$, $\bar{P}_1$ is a normal subgroup of $\bar{\mathrm{C}}'(p) \cap \bar{\mathrm{C}}(p)$ with index 3.

We shall prove in the next section that the symmetry group G_{sym} of a group covariant SIC POVM is a subgroup of some Clifford group for any prime dimension. It follows from the above discussion that the number of HW groups in G_{sym} may only take three possible values $1, p, p^2$; that is, the SIC POVM may only be covariant with respect to $1, p$ or p^2 HW groups. If $|G_{\mathrm{sym}}|$ is not divisible by p^4, then G_{sym} contains only one HW group. Otherwise, each Sylow p-subgroup of G_{sym} is also a Sylow p-subgroup of the Clifford group containing G_{sym}. In addition, the number of Sylow p-subgroups in G_{sym} is either 1 or $p+1$, according to Sylow's theorem [26], so the number of HW groups in G_{sym} is either p or p^2.

3. The symmetry group of any group covariant SIC POVM in any prime dimension is a subgroup of certain Clifford group

In this section, we prove that the (extended) symmetry group of any group covariant SIC POVM in any prime dimension is a subgroup of certain (extended) Clifford group, and derive simple criteria on determining whether two group covariant SIC POVMs are unitarily or antiunitarily equivalent. First, we show that a group covariant SIC POVM in any prime dimension is covariant with respect to the HW group. Then we prove our main result in three cases separately, namely the special case $p = 2$, the general case $p \geq 5$, and the special case $p = 3$.

3.1. A group covariant SIC POVM in any prime dimension is covariant with respect to the Heisenberg–Weyl group

In this section, we prove the following lemma.

Lemma 1 *In any prime dimension, a group covariant SIC POVM is necessarily covariant with respect to the HW group.*

First, we need the following proposition.

Proposition 2 *In any finite dimension, if a SIC POVM is covariant with respect to a unimodular unitary group G, then G is necessarily nonabelian. In particular, the symmetry group of any group covariant SIC POVM is necessarily nonabelian.*

Suppose there exists a SIC POVM in dimension d which is covariant with respect to an abelian unimodular unitary group G, which may be assumed to be diagonal, without loss of generality. Let $|\psi_j\rangle = (a_{j1}, \ldots, a_{jd})^T$ for $j = 1, \ldots, d^2$ be the d^2 vectors in the SIC POVM. Since $|\psi_j\rangle$s are related to each other by the diagonal unitary group G, the modulus of a given entry is independent of the vectors; that is, $|a_{jk}|$ is independent of j. Hence, the condition $\sum_{j=1}^{d^2} |\psi_j\rangle\langle\psi_j| = dI$ implies that $|a_{jk}| = 1/\sqrt{d}, \forall j, k$. Suppose $U = \mathrm{diag}(u_1, \ldots, u_d)$ is an element in G which does not stabilize $|\psi_1\rangle$; then we have, according to (1),

$$d^2|\langle\psi_1|U|\psi_1\rangle|^2 = \left|\sum_{j=1}^{d} u_j\right|^2 = \frac{d^2}{d+1}. \tag{26}$$

Since u_js are roots of unity, which are algebraic integers, the expression in the middle of the above equation is also an algebraic integer. On the other hand, $d^2/(d+1)$ cannot be an algebraic integer since it is a fraction which is not an integer [30], a contradiction. This completes the proof of the first part of proposition 2. The second part of proposition 2 follows immediately, since a SIC POVM is group covariant if and only if it is covariant with respect to its symmetry group.

Let us now turn back to the proof of lemma 1. In any prime dimension, a group covariant SIC POVM is also necessarily covariant with respect to any Sylow p-subgroup, say P, of its symmetry group G_{sym}. Since P must be nonabelian according to proposition 2, the order of P is at least p^3; recall that all groups of order p or p^2 are abelian. In addition, P must be irreducible when taken as a representation of itself, because the degree of any irreducible representation of a finite group divides its order [30]. The centre of P has order at least p because a p-group has a nontrivial centre [26]. Since P is irreducible, any element in its centre is proportional to the identity, which, together with the unimodular condition, implies that the centre is the cyclic group $\langle\omega I\rangle$ generated by ωI. Since the p-group $P/\langle\omega I\rangle$ also has a nontrivial centre, there exists a nontrivial element X' in P such that $X'\langle\omega I\rangle$ commutes with all elements in $P/\langle\omega I\rangle$. Hence, there exists another element Z' in P, such that $Z'X'Z'^{-1}X'^{-1} = \omega^k I$ with $1 \leq k < p$. In addition, k can be chosen to be 1.

Since any irreducible representation of a p-group is monomial [31], we can assume that P is in monomial form, without loss of generality. Note that at least one of the two elements, say Z', is not diagonal, and that each element of P which is not diagonal necessarily has the same spectrum as that of Z. We can choose a new basis such that $Z' = \mathrm{diag}(1, \omega, \ldots, \omega^{p-1})$, while leaving the commutation relation between Z' and X' unchanged. It follows that $X'X^{-1}$ commutes with Z', and is thus diagonal. We can turn X' into X with a suitable diagonal unitary transformation which leaves Z' unchanged. Hence, the subgroup of P generated by the two elements X' and Z' is unitarily equivalent to the HW group defined in (2). We may now identify X' and Z'

with X and Z respectively, and call the group generated by the two elements HW group. It remains to show that the SIC POVM is covariant with respect to this HW group, which is guaranteed if, for each vector in the SIC POVM, the stability group within this HW group does not contain any nontrivial element. Suppose otherwise, without loss of generality, we can assume that a vector $|\psi\rangle$ of the SIC POVM is stabilized by Z. Then $|\psi\rangle$ can only have one nonzero entry, which implies that $X|\psi\rangle$ and $|\psi\rangle$ are orthogonal to each other, a contradiction. This completes the proof of lemma 1. Hence, for any SIC POVM in any prime dimension, group covariance is equivalent to HW covariance.

3.2. Special case p=2

In dimension two, the Clifford group $\bar{\mathrm{C}}(2)$ is generated by the Hadamard operator $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and phase operator $\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix}$ [32], and the extended Clifford group is generated by the complex conjugation operator $\hat{J}$ in addition to the two operators. The orders of the Clifford group and the extended Clifford group are 24 and 48 respectively. There is only one orbit of fiducial vectors under either the Clifford group or the extended Clifford group. One of the fiducial vectors is given by

$$|\psi\rangle = \begin{pmatrix} \sqrt{\frac{3+\sqrt{3}}{6}} \\ \mathrm{e}^{\mathrm{i}\pi/4}\sqrt{\frac{3-\sqrt{3}}{6}} \end{pmatrix}. \tag{27}$$

The order of the stability group of each fiducial vector within the Clifford group (extended Clifford group) is 3 (6); thus there are eight fiducial vectors constituting two SIC POVMs [1–3].

When represented on the Bloch sphere, the eight fiducial vectors constitute a cube, and the two SIC POVMs constitute two regular tetrahedra respectively, which are related to each other by space inversion. The Clifford group corresponds to the rotational symmetry group of the cube, while the extended Clifford group corresponds to the full symmetry group of the cube. The extended symmetry group of each SIC POVM corresponds to the full symmetry group of the tetrahedron, which is a subgroup of the full symmetry group of the cube. Hence, the extended symmetry group of each SIC POVM contains only one HW group, and it is a subgroup of the extended Clifford group.

Moreover, all SIC POVMs in dimension two are unitarily equivalent, since any SIC POVM, when represented on the Bloch sphere, corresponds to a regular tetrahedron. Hence, any SIC POVM in dimension two is covariant with respect to a unique HW group, and its (extended) symmetry group is a subgroup of the (extended) Clifford group.

3.3. General case $p \geq 5$

For $p \geq 5$, the following theorem proved by Sibley [33] is crucial to our later discussion.

Theorem 3 (Sibley) *Suppose G is a finite group with a faithful, irreducible, unimodular and quasiprimitive representation of prime degree $p \geq 5$. If a Sylow p-subgroup P of G has order p^3, then P is normal in G, and G/P is isomorphic to a subgroup of* $\mathrm{SL}(2,p)$.

A quasiprimitive representation is one whose restriction to every subgroup is homogeneous, that is a multiple of one irreducible representation of the subgroup. An irreducible representation of prime degree that is not quasiprimitive is monomial [33].

Let $G_{\rm sym}$ be the symmetry group of a group covariant SIC POVM in any prime dimension $p \geq 5$. Taken as a representation of itself, $G_{\rm sym}$ is irreducible, because it contains the HW group D, which is irreducible. To apply Sibley's theorem, we shall first prove that the order of each Sylow p-subgroup of $G_{\rm sym}$ is p^3. Suppose otherwise, then the HW group D is a proper subgroup of one of the Sylow p-subgroups. The normalizer $N(D)$ of D in this Sylow p-subgroup is strictly larger than D, and $N(D)/D$ contains a subgroup of order p [26]. It follows that $N(D)$ contains a subgroup of order p^4 which in turn contains D as a normal subgroup. This group of order p^4 is also a Sylow p-subgroup of the Clifford group, and can be taken as P_1 without loss of generality, since all Sylow p-subgroups are conjugated to each other. Suppose $V \in \left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \mathbf{0}\right]$ is unimodular; then the group $H = \langle \omega I, Z, V \rangle$ is a normal subgroup of P_1 consisting of its diagonal elements. Since the order of the stability group within H of each fiducial vector is the same, and the action cannot be transitive according to proposition 2, the fiducial vectors form p orbits of equal length p. It follows that each fiducial vector, say $|\psi\rangle$, is stabilized by some nontrivial element of the form $V^j Z^k$, with $1 \leq j \leq p-1$ and $0 \leq k \leq p-1$. Since $V^j Z^k$ is diagonal, all p fiducial vectors $Z^l|\psi\rangle$ for $l = 0, \ldots, p-1$ are simultaneously stabilized by it. These fiducial vectors must belong to a same eigenspace of $V^j Z^k$. However, each eigenspace of $V^j Z^k$ for $1 \leq j \leq p-1$ has dimension of at most two as shown in section 2.3, and thus cannot admit more than two fiducial vectors when $p \geq 5$; hence a contradiction would arise. In conclusion, the order of each Sylow p-subgroup of $G_{\rm sym}$ is p^3, and D is a Sylow p-subgroup of $G_{\rm sym}$.

If $G_{\rm sym}$ is quasiprimitive, then D is a normal subgroup of $G_{\rm sym}$, or equivalently, $G_{\rm sym}$ is a subgroup of the Clifford group of D, according to Sibley's theorem. In addition, $G_{\rm sym}$ contains only one HW group.

Now suppose $G_{\rm sym}$ is in monomial form. By a suitable monomial unitary transformation and a different choice of generators if necessary, we can always keep the HW group in the standard form. Let T be the normal subgroup of $G_{\rm sym}$ which consists of its diagonal elements, and $S = G_{\rm sym}/T$. Note that $|S|$ is divisible by p, otherwise the SIC POVM would be covariant with respect to the abelian group T, which contradicts proposition 2. Hence, $|T|$ is not divisible by p^3, since $|G_{\rm sym}|$ is not divisible by p^4 as shown previously. In addition, under the action of T, the fiducial vectors of the SIC POVM form p orbits of equal length p, and two fiducial vectors are on the same orbit generated by T if and only if they are on the same orbit generated by $\langle Z \rangle$.

To show that D is a normal subgroup of G_{sym}, we shall first show that T contains no other elements except those generated by Z and ωI; that is, $T=\langle\omega I, Z\rangle$ with $|T|=p^2$. Suppose $|T|>p^2$, then T contains a nontrivial element U whose order is not a multiple of p. Since U cannot stabilize all fiducial vectors, there exists at least one fiducial vector, say $|\psi\rangle$, not stabilized by U. On the other hand, since $U|\psi\rangle$ and $|\psi\rangle$ are on the same orbit of T and hence on the same orbit of $\langle Z\rangle$, we have $U|\psi\rangle=\mathrm{e}^{\mathrm{i}\phi}Z^k|\psi\rangle$, where $1\leq k\leq p-1$, and $\mathrm{e}^{\mathrm{i}\phi}$ is an overall phase factor. Note that $|\psi\rangle$ has at least two nonzero entries; let $\mathrm{e}^{\mathrm{i}\phi_1}, \mathrm{e}^{\mathrm{i}\phi_2}$ be the two diagonal entries of U corresponding to any two nonzero entries of $|\psi\rangle$ respectively; then $\mathrm{e}^{\mathrm{i}(\phi_1-\phi_2)}$ is a primitive pth root of unity, contradicting the fact that the order of U is not a multiple of p.

Now we are ready to show that D is a normal subgroup of G_{sym}. Let U be an arbitrary element in G_{sym}. Since $T=\langle\omega I, Z\rangle$ is a normal subgroup of G_{sym}, $UZU^\dagger=\omega^{k_1}Z^{k_2}$ for some integers k_1, k_2, it remains to show that $UXU^\dagger\in D$, or equivalently $U^\dagger XU\in D$. According to the following equalities:

$$U^\dagger XUZ(U^\dagger XU)^\dagger=U^\dagger X\omega^{k_1}Z^{k_2}X^\dagger U=U^\dagger\omega^{k_1-k_2}Z^{k_2}U=\omega^{-k_2}Z, \qquad (28)$$

$X^{-k_2}U^\dagger XU$ commutes with Z, and hence belongs to T, which in turn implies that $X^{-k_2}U^\dagger XU=\omega^{k_1'}Z^{k_2'}$ for some integers k_1', k_2', and $U^\dagger XU=\omega^{k_1'}X^{k_2}Z^{k_2'}\in D$. Hence, when G_{sym} is monomial, G_{sym} is also a subgroup of the Clifford group, and contains only one HW group. Moreover, in this case, the stability group of each fiducial vector is cyclic, and there exists a fiducial vector on the same orbit of the Clifford group whose stability group is generated by a permutation matrix. To demonstrate this point, note that the stability group of each fiducial vector is isomorphic to G_{sym}/D, and its order is not a multiple of p. Since G_{sym} is monomial, G_{sym}/D is isomorphic to a subgroup of N_1 according to the discussions in section 2.3. It follows that the stability group of each fiducial vector is cyclic; recall that all subgroups of N_1 whose order is not a multiple of p are cyclic. In addition, the generator of the stability group is conjugated to the class representative $[a^k, \mathbf{0}]$ with $0\leq k\leq\frac{p-1}{2}$ (the isomorphism image of the class representative $(a^k, \mathbf{0})$ listed in table 2). According to section 2.3, $[a^k, \mathbf{0}]$ is a permutation matrix up to an overall phase factor, so there exists a fiducial vector whose stability group is generated by a permutation matrix. This observation may help search for SIC POVMs with certain specific symmetry.

In conclusion, for any prime dimension larger than three, each group covariant SIC POVM is covariant under one and only one HW group, and its symmetry group is a subgroup of the Clifford group. The conclusion can also be extended to cover antiunitary operations. Note that any antiunitary operation in the extended symmetry group of the SIC POVM must preserve the HW group, and thus must belong to the extended Clifford group. Also, the same results hold for dimension two according to the discussion in section 3.2. So we obtain

Theorem 4 *In any prime dimension not equal to three, each group covariant SIC POVM is covariant with respect to a unique HW group. Furthermore, its (extended) symmetry group is a subgroup of the (extended) Clifford group.*

To determine whether two given SIC POVMs are equivalent can be a very challenging task because there are infinitely many unitary (antiunitary) operations to test. Usually, we need to find the explicit transformation to claim that they are equivalent, and we need to find some invariant that can distinguish the two SIC POVMs to claim that they are inequivalent. These two approaches will be illustrated with SIC POVMs in dimension three in section 4. However, neither approach is easy in general. The difficulty is reflected in the following long-standing open question: Are two SIC POVMs on two different orbits of the (extended) Clifford group equivalent? Fortunately, we can solve this open question for prime dimensions not equal to three in virtue of theorem 4.

Consider two HW covariant SIC POVMs in any prime dimension not equal to three. If there exists a unitary (antiunitary) operation which maps one SIC POVM to the other, it must preserve the HW group, and hence belongs to the Clifford group (extended Clifford group). It follows that the two SIC POVMs are on the same orbit of the Clifford group (extended Clifford group). So we obtain

Corollary 5 *In any prime dimension not equal to three, two SIC POVMs covariant with respect to the same HW group are unitarily (unitarily or antiunitarily) equivalent if and only if they are on the same orbit of the Clifford group (extended Clifford group).*

In any prime dimension not equal to three, as an immediate consequence of theorem 4 and corollary 5, the different orbits of SIC POVMs found by Scott and Grassl [4] are not unitarily or antiunitarily equivalent. In particular, two HW covariant SIC POVMs cannot be unitarily or antiunitarily equivalent if their respective fiducial vectors have non-isomorphic stability groups (within the extended Clifford group). For example, the two orbits of SIC POVMs in dimension seven discovered by Appleby [3] are not unitarily or antiunitarily equivalent. However, it should be emphasized that, without theorem 4 and corollary 5, this seemingly obvious criterion is not well justified *a priori*. We shall see counterexamples in dimension three in section 4.1.

3.4. Special case $p = 3$

Now consider the special case $p = 3$. First, assume that G_{sym} is not monomial. According to the classification of finite linear groups of degree 3 by Blichfeldt [24], the order of the Sylow p-subgroup of G_{sym} is at most p^4, and if it is equal to p^4, then G_{sym} is isomorphic to some subgroup of the Clifford group. If the order of the Sylow p-subgroup is p^3, there is a counterexample to Sibley's theorem (theorem 3). This counterexample is a unimodular unitary group of order 1080 whose collineation group (of order 360) is isomorphic to the alternating group of six letters [24]. However, this group cannot be the symmetry group of any SIC POVM. Suppose there exists a SIC POVM with this group as its symmetry group; let U be an order 5 element in the group. Under the action of the group generated by U, the nine fiducial vectors form disjoint orbits of length either 1 or 5. It follows that there are four orbits of length 1, that is, four fiducial vectors stabilized by U. These four fiducial vectors must belong to a same eigenspace

of U; otherwise at least two of them would be orthogonal to each other. On the other hand, the dimension of this eigenspace is at most two, because U is not proportional to the identity. However, a two-dimensional subspace cannot admit four fiducial vectors. Suppose otherwise, from the Bloch sphere representation, one can see that the maximum pairwise fidelity among the four fiducial vectors is no smaller than $\frac{1}{3}$, contradicting the fact that the pairwise fidelity among fiducial vectors of a SIC POVM in dimension three is equal to $\frac{1}{4}$. In conclusion, $G_{\rm sym}$ must be a subgroup of some Clifford group when it is not monomial.

Now suppose that $G_{\rm sym}$ is in monomial form, and that one of the HW groups contained in $G_{\rm sym}$ is in the standard form, as in the case $p \geq 5$. Let T be the normal subgroup of $G_{\rm sym}$ consisting of its diagonal elements, and $S = G_{\rm sym}/T$.

If $T = \langle \omega I, Z \rangle$, then we can conclude that $G_{\rm sym}$ contains only one HW group, and it is a subgroup of the Clifford group, following a similar reasoning as that applied to the case $p \geq 5$. However, it turns out that this situation does not occur for the special case $p = 3$ [3] (see also section 4), in sharp contrast with the general case $p \geq 5$.

Otherwise, each fiducial vector is stabilized by some nontrivial element in T. Let $|\psi\rangle$ be a fiducial vector, and U a nontrivial element in T that stabilizes $|\psi\rangle$. Simple analysis shows that two of the diagonal entries of U must be identical, and $|\psi\rangle$ must have two nonzero entries with equal modulus $\frac{1}{\sqrt{2}}$ and a zero entry. With out loss of generality, we may assume that $U = \mathrm{e}^{\mathrm{i}\phi'}\mathrm{diag}(1, 1, \mathrm{e}^{\mathrm{i}\phi})$, and $|\psi\rangle = \frac{1}{\sqrt{2}}(1, \mathrm{e}^{\mathrm{i}t}, 0)$; indeed all vectors of this form are fiducial vectors [3] (see also section 4). To ensure that $UX|\psi\rangle$ be a fiducial vector in the SIC POVM, ϕ can only take two possible values $\pm\frac{2\pi}{3}$. We can choose $U = \mathrm{e}^{-\mathrm{i}2\pi/9}\mathrm{diag}(1, 1, \omega)$ for definiteness, where ϕ' has been chosen such that U is unimodular. Now it is straightforward to verify that T cannot contain any elements other than those generated by the following three elements, $\omega I, Z, U$ and $|T| = 27$.

The order of the group S may either be 3 or 6 and, correspondingly, the order of $G_{\rm sym}$ is either 81 or 162. If $|S| = 3$, then $G_{\rm sym}$ is a p-group of order 3^4; hence the normalizer of D in $G_{\rm sym}$ is strictly larger than D [26], which implies that D is a normal subgroup of $G_{\rm sym}$. If $|S| = 6$, $G_{\rm sym}$ contains a Sylow p-subgroup P of order 81 and with index 2, such that D is a normal subgroup of P. Note that P is also a Sylow p-subgroup of the Clifford group, and thus contains two other normal subgroups which are unitarily equivalent to D, or two other HW groups, as shown in Sec 2.4. At least one of the three HW groups is also a normal subgroup of $G_{\rm sym}$. In fact, according to our discussion in section 2.4, only one of the three HW groups is normal in $G_{\rm sym}$, and the other two are conjugated to each other. Hence, when $G_{\rm sym}$ is monomial, $G_{\rm sym}$ is also a subgroup of some Clifford group. In addition, in this case, the stability group of each fiducial vector is cyclic, because, according to section 2.3, $G_{\rm sym}/D$ is isomorphic to a subgroup of N_1, which is cyclic.

In conclusion, the symmetry group of any group covariant SIC POVM in dimension three is also a subgroup of some Clifford group. However, it should be emphasized that a SIC POVM may be covariant under more than one HW groups. Furthermore, its symmetry group may be a subgroup of the Clifford group of one of the HW groups but

not a subgroup of other Clifford groups, say the standard Clifford group.

According to section 2.4, the symmetry group $\bar{G}_{\mathrm{sym}}$ of a SIC POVM may contain three or nine HW groups if it contains more than one. The order of $\bar{G}_{\mathrm{sym}}$ may only take three possible values 27, 54 or 216, and cannot take the value 108, since the quotient group of $\bar{G}_{\mathrm{sym}}$ with respect to its normal HW subgroup is isomorphic to a subgroup of $\mathrm{SL}(2,3)$, which has no subgroup of order 12 [34]. Let $\bar{D}^{(1)},\ldots,\bar{D}^{(k)}$ be the HW groups contained in $\bar{G}_{\mathrm{sym}}$, and $\bar{\mathrm{C}}^{(1)}(p),\ldots,\bar{\mathrm{C}}^{(k)}(p)$ their associated Clifford groups respectively, where $k=3$ or $k=9$. The symmetry group of the SIC POVM within the Clifford group $\bar{\mathrm{C}}^{(j)}(p)$ is $\bar{G}^{(j)}_{\mathrm{sym}}=\bar{G}_{\mathrm{sym}}\cap\bar{\mathrm{C}}^{(j)}(p)$. According to the previous discussions, at least one of them is identical with $\bar{G}_{\mathrm{sym}}$, and each $\bar{G}^{(j)}_{\mathrm{sym}}$ different from $\bar{G}_{\mathrm{sym}}$ is a p-group of order 27, which is isomorphic to the Sylow p-subgroup $\bar{P}_1$ of the standard Clifford group, and thus contains three HW groups (see section 2.4). Moreover, each $\bar{G}^{(j)}_{\mathrm{sym}}$ contains at least one HW group whose associated Clifford group contains $\bar{G}_{\mathrm{sym}}$. In virtue of this observation, we can easily determine the symmetry group of any group covariant SIC POVM in dimension three, no matter whether it is a subgroup of the standard Clifford group.

If $|\bar{G}_{\mathrm{sym}}|=27$, $\bar{G}_{\mathrm{sym}}$ is the intersection of the three Clifford groups associated with the three HW groups contained in $\bar{G}_{\mathrm{sym}}$ respectively; in other words, $\bar{G}^{(1)}_{\mathrm{sym}},\bar{G}^{(2)}_{\mathrm{sym}},\bar{G}^{(3)}_{\mathrm{sym}}$ all coincides with $\bar{G}_{\mathrm{sym}}$. Hence, starting from any HW group contained in $\bar{G}_{\mathrm{sym}}$, the symmetry group within its Clifford group is the same. If $|\bar{G}_{\mathrm{sym}}|=54$, $\bar{G}_{\mathrm{sym}}$ also contains three HW groups; however, only one of the three groups $\bar{G}^{(1)}_{\mathrm{sym}},\bar{G}^{(2)}_{\mathrm{sym}},\bar{G}^{(3)}_{\mathrm{sym}}$ is identical with $\bar{G}_{\mathrm{sym}}$. That is, $\bar{G}_{\mathrm{sym}}$ is a subgroup of only one of the Clifford groups associated with the three HW groups respectively. If $|\bar{G}_{\mathrm{sym}}|=216$, $\bar{G}_{\mathrm{sym}}$ contains nine HW groups, and is also a subgroup of only one of the Clifford groups of these HW groups respectively. In either of the later two cases, starting from different HW groups, we may "see" different symmetry groups, if we only consider symmetry operations within the Clifford group of the given HW group.

Now it is straightforward to extend the above analysis to show that the extended symmetry group $\overline{EG}_{\mathrm{sym}}$ of a group covariant SIC POVM is a subgroup of some extended Clifford group. Suppose the extended symmetry group of the SIC POVM contains antiunitary operations (otherwise the claim is already proved); then $\bar{G}_{\mathrm{sym}}$ is a normal subgroup of $\overline{EG}_{\mathrm{sym}}$ with index 2. If $|\bar{G}_{\mathrm{sym}}|=27$, $\bar{G}_{\mathrm{sym}}$ contains three HW groups all of which are normal. At least one of the three HW groups is also normal in $\overline{EG}_{\mathrm{sym}}$, and $\overline{EG}_{\mathrm{sym}}$ is a subgroup of the extended Clifford group of this HW group. If $|\bar{G}_{\mathrm{sym}}|=54$, $\bar{G}_{\mathrm{sym}}$ contains three HW groups one of which is normal, and the other two are conjugated to each other. The normal HW group in $\bar{G}_{\mathrm{sym}}$ must remain normal in $\overline{EG}_{\mathrm{sym}}$, otherwise all three HW groups would be conjugated to each other in $\overline{EG}_{\mathrm{sym}}$, which contradicts the fact that the index of $\bar{G}_{\mathrm{sym}}$ in $\overline{EG}_{\mathrm{sym}}$ is 2. The same analysis is also applicable when $|\bar{G}_{\mathrm{sym}}|=216$. So we obtain

Theorem 6 *In dimension three, each group covariant SIC POVM may be covariant with respect to three or nine HW groups, and its symmetry group within the standard Clifford group contains at least three HW groups. Furthermore, the (extended) symmetry*

group of the SIC POVM is a subgroup of at least one of the (extended) Clifford groups associated with these HW groups respectively.

In dimension three, not surprisingly, there are counterexamples to corollary 5 in section 3.3, since a SIC POVM may be covariant under more than one HW groups. If a unitary operation maps an HW covariant SIC POVM to another HW covariant one, then it must map one of the HW groups $D^{(1)}, \ldots, D^{(k)}$ to the standard HW group D. Let $U^{(j)}$ be a unitary transformation that maps $D^{(j)}$ to D. By applying two transformations $U^{(j)}, U^{(l)}$ with $1 \le j, l \le k$ respectively to a given HW covariant SIC POVM, two other HW covariant SIC POVMs are obtained. These two SIC POVMs are on the same orbit of the standard Clifford group if and only if $D^{(j)}, D^{(l)}$ are conjugated to each other in G_{sym}. Hence, for each HW covariant SIC POVM in dimension three, the number of orbits of unitarily equivalent SIC POVMs is equal to the number of conjugacy classes of the HW groups contained in the symmetry group G_{sym}.

In addition, two SIC POVMs are on the same orbit of the Clifford group if and only if they are on the same orbit of the extended Clifford group. To demonstrate this point, without loss of generality, we may assume that G_{sym} contains the Sylow p-subgroup P_1 of the Clifford group. Then each fiducial vector of the SIC POVM can be written in the form $\frac{1}{\sqrt{2}}(1, \mathrm{e}^{\mathrm{i}t}, 0)$ up to permutations of the three entries. Since the symmetry group of any SIC POVM in this family contains antiunitary operations, the orbit length in the Clifford group is the same as that in the extended Clifford group [3] (see also section 4). So we obtain

Corollary 7 *In dimension three, for each SIC POVM covariant with respect to the HW group, there are three orbits (both under the Clifford group and the extended Clifford group) of equivalent SIC POVMs if its symmetry group has order 27, and two orbits if the symmetry group has order 54 or 216. In either case, the orbits of equivalent SIC POVMs are connected to each other by unitary transformations that map additional HW groups contained in the symmetry group within the standard Clifford group to the standard HW group.*

4. SIC POVMs in three-dimensional Hilbert space

SIC POVMs in dimension three are special in many aspects. A SIC POVM may be covariant with respect to more than one HW groups; SIC POVMs on different orbits may be equivalent even if their fiducial vectors have stability groups (within the standard Clifford group) of different orders; in particular, there exist continuous orbits of fiducial vectors. In this section, we establish a complete equivalence relation of SIC POVMs on different orbits in dimension three, and classify inequivalent ones according to the geometric phases associated with fiducial vectors. In addition, we show that additional SIC POVMs can be constructed by regrouping of the fiducial vectors from different SIC POVMs which may or may not be on the same orbit of the extended Clifford group. The methods used for SIC POVMs in dimension three are also applicable to SIC POVMs in other dimensions, no matter prime or not.

4.1. Infinitely many inequivalent SIC POVMs

There is a one-parameter family of fiducial vectors in dimension three,

$$|\psi_{\mathrm{f}}(t)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -\mathrm{e}^{\mathrm{i}t} \end{pmatrix}, \tag{29}$$

and for each distinct orbit, there is exactly one value of $t \in [0, \frac{\pi}{3}]$, such that $|\psi_{\mathrm{f}}(t)\rangle$ is on the orbit. There are three kinds of orbits, two exceptional orbits corresponding to the endpoints $t = 0$ and $t = \frac{\pi}{3}$ respectively, and infinitely many generic orbits corresponding to $0 < t < \frac{\pi}{3}$ [1–3].

According to Appleby [3], the order of the stability group within the Clifford group (extended Clifford group) of each fiducial vector is 24, 6, 3 (48, 12, 6) for the three kinds of orbits respectively. Hence, the number of SIC POVMs on each orbit is 1, 4, 8 respectively. The orbit length is the same under both the Clifford group and the extended Clifford group. The stability group (within the extended Clifford group) of the exceptional vector $|\psi_{\mathrm{f}}(0)\rangle$ consists of all operations of the form $[F, \mathbf{0}]$ with $F \in \mathrm{ESL}(2, 3)$. The stability group of the exceptional vector $|\psi_{\mathrm{f}}(\frac{\pi}{3})\rangle$ is generated by the unitary operation

$$[F, \chi] = \left[\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] \tag{30}$$

and antiunitary operation

$$[A, \chi] = \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]. \tag{31}$$

For a generic vector $|\psi_{\mathrm{f}}(t)\rangle$ with $0 < t < \frac{\pi}{3}$, the stability group is generated by the unitary operation

$$[F, \chi]^2 = \left[\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right] \tag{32}$$

and antiunitary operation

$$[F, \chi] \circ [A, \chi] = \left[\begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right]. \tag{33}$$

For each generic vector, the stability group is independent of t; in particular, the two fiducial vectors $\hat{J}|\psi_{\mathrm{f}}(t)\rangle = |\psi_{\mathrm{f}}(-t)\rangle$ and $|\psi_{\mathrm{f}}(t)\rangle$ have the same stability group, where $\hat{J}$ is the complex conjugation operator (see section 2.3).

For each generic orbit, the eight SIC POVMs on the orbit form four pairs $[F_k, \mathbf{0}]|\psi_{\mathrm{f}}(t)\rangle, [F_k, \mathbf{0}]\hat{J}|\psi_{\mathrm{f}}(t)\rangle$ (here each fiducial vector represents the SIC POVM containing it) for $k = 1, 2, 3, 4$, where

$$F_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad F_2 = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \qquad F_3 = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \qquad F_4 = \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}. \tag{34}$$

Within the Clifford group, the two SIC POVMs in each pair share the same symmetry group, which is a Sylow 3-subgroup of the Clifford group. There is a one-to-one correspondence between the four pairs of SIC POVMs and the four Sylow 3-subgroups of the Clifford group. The symmetry group of each SIC POVM contains three HW groups; all of which are normal subgroups. The intersection of any two symmetry groups of two SIC POVMs from different pairs respectively is exactly the standard HW group. For the exceptional orbit corresponding to $t = \frac{\pi}{3}$, each SIC POVM is invariant under the complex conjugation operation, so the two SIC POVMs in each pair merge to one. The symmetry group of each SIC POVM is the normalizer (within the Clifford group) of a Sylow 3-subgroup, and contains three HW groups too. However, only the standard HW group is a normal subgroup, and the other two are conjugated to each other. The intersection of any two symmetry groups is the group generated by $[-\mathbf{1}, \mathbf{0}]$ and the standard HW group. For the exceptional orbit corresponding $t = 0$, there is only one SIC POVM, and its symmetry group is the Clifford group, which contains nine HW groups. The standard HW group is a normal subgroup, while the other eight HW groups are conjugated to each other.

We are now ready to show the equivalence relation of SIC POVMs among different orbits in virtue of theorem 6 and corollary 7 derived in section 3.4. Note that for each $t \in [0, \frac{\pi}{3}]$, the symmetry group of the SIC POVM generated from the fiducial vector $|\psi_{\mathrm{f}}(t)\rangle$ contains as a subgroup the Sylow 3-subgroup P_1 of the Clifford group. According to (22) and (23), $U = \mathrm{diag}(1, \mathrm{e}^{-2\mathrm{i}\pi/9}, \mathrm{e}^{-4\mathrm{i}\pi/9})$ is a unitary transformation that permutes the three HW groups contained in P_1. According to corollary 7, the SIC POVMs on the three orbits generated from $|\psi_{\mathrm{f}}(t)\rangle$, $U^\dagger|\psi_{\mathrm{f}}(t)\rangle$ and $U^{\dagger 2}|\psi_{\mathrm{f}}(t)\rangle$ respectively are unitarily equivalent. That is, SIC POVMs on the three orbits corresponding to the parameters $t, \frac{2\pi}{9}+t, \frac{2\pi}{9}-t$ respectively for each $t \in [0, \frac{\pi}{9}]$ (when $t = 0$ or $t = \frac{\pi}{9}$, two of the three orbits may merge) are unitarily equivalent. Moreover, SIC POVMs on any two different orbits corresponding to $t \in [0, \frac{\pi}{9}]$ are not equivalent. Hence, there are two orbits of equivalent SIC POVMs for each exceptional orbit, and three orbits of equivalent SIC POVMs for each generic orbit with $t \neq \frac{\pi}{9}, \frac{2\pi}{9}$.

The equivalence of the exceptional orbit with $t = 0$ and the generic orbit with $t = \frac{2\pi}{9}$ is particularly surprising at first glance, since they have stability groups of different orders (within the standard Clifford group). Equally surprising is the equivalence of the exceptional orbit with $t = \frac{\pi}{3}$ and the generic orbit with $t = \frac{\pi}{9}$.

In addition, the (extended) symmetry group of any SIC POVM except those on the orbit with $t = \frac{\pi}{9}$ or $t = \frac{2\pi}{9}$ is a subgroup of the standard (extended) Clifford group. For each SIC POVM on the orbit with $t = \frac{\pi}{9}$ or $t = \frac{2\pi}{9}$, its (extended) symmetry group is a subgroup of the (extended) Clifford group associated with another HW group contained in the symmetry group within the standard Clifford group.

In conjunction with theorem 4, we obtain a quite surprising conclusion: Among all (HW) group covariant SIC POVMs in prime dimensions, the SIC POVMs in dimension three on the orbits generated from the fiducial vectors in (29) with $t = \frac{\pi}{9}$ and $t = \frac{2\pi}{9}$ respectively are the only ones whose (extended) symmetry groups are not subgroups of

the standard (extended) Clifford group.

If we denote by $\mathrm{A}_\pm$, $\mathrm{B}_\pm$, $\mathrm{C}_\pm$ the six SIC POVMs (two on each of the three orbits of equivalent SIC POVMs) containing the fiducial vectors $|\psi_{\mathrm{f}}(\pm t)\rangle$, $|\psi_{\mathrm{f}}(\pm(\frac{2\pi}{9}-t))\rangle$, $|\psi_{\mathrm{f}}(\pm(\frac{2\pi}{9}+t))\rangle$ respectively, then the transformation among the six SIC POVMs induced by $U^\dagger$ can be illustrated as follows:

$$\mathrm{A}_+ \rightarrow \mathrm{C}_+ \rightarrow \mathrm{B}_-, \qquad \mathrm{A}_- \rightarrow \mathrm{B}_+ \rightarrow \mathrm{C}_-. \tag{35}$$

Interestingly, the three SIC POVMs $\mathrm{A}_+, \mathrm{B}_-, \mathrm{C}_+$ cycles among the three obits in the opposite direction compared with the other three SIC POVMs $\mathrm{A}_-, \mathrm{B}_+, \mathrm{C}_-$. Although the SIC POVMs on the three orbits with t, $\frac{2\pi}{9}-t$, $\frac{2\pi}{9}+t$ respectively are equivalent, the orbits themselves are not equivalent in the sense that there is no unitary or antiunitary transformation that can map all SIC POVMs or fiducial vectors on one of the three orbits to that on another one. For example, under the transformation induced by $U^\dagger$, only six out of the 24 SIC POVMs on the three orbits are permuted among each other; the other 18 SIC POVMs are no longer on any of the three orbits. This point will become more clear when we study the additional SIC POVMs constructed by regrouping of the fiducial vectors in section 4.2.

To better characterize those inequivalent SIC POVMs, we need to find some invariants that can distinguish them. The simplest invariant involves three different vectors in a SIC POVM. Let $\rho_j = |\psi_j\rangle\langle\psi_j|$ for $j = 1,2,3$, where $|\psi_j\rangle$s are three different vectors in a SIC POVM. The trace of the triple product $\mathrm{tr}(\rho_1\rho_2\rho_3)$ is invariant under unitary transformation. This invariant has been applied by Appleby *et al.* [16] to studying the set of occurring probabilities of measurement outcomes of SIC POVMs. According to (1), $|\mathrm{tr}(\rho_1\rho_2\rho_3)| = \frac{1}{8}$ for $d = 3$, so the relevant invariant is the phase of the trace, $\phi' = \arg[\mathrm{tr}(\rho_1\rho_2\rho_3)]$, with $-\pi \leq \phi' < \pi$. Since odd permutations or complex conjugation of the three states reverses the sign of the phase, we shall be concerned with the absolute value of the phase, $\phi = |\arg[\mathrm{tr}(\rho_1\rho_2\rho_3)]|$, with $0 \leq \phi \leq \pi$. Now ϕ is independent of the permutations and complex conjugation of the three states. Recall that the phase ϕ defined above is exactly the discrete geometric phase associated with the three vectors $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle$, which is the additional phase appearing after traversing the geodesic triangle with the three vectors as vertices in the projective Hilbert space [21,22]. It is also known as the Bargmann invariant [23]. Another invariant associated with the three vectors is the set of eigenvalues of the sum $M = \rho_1 + \rho_2 + \rho_3$. However, some simple algebra shows that it is not an independent invariant:

$$\begin{aligned}
&\mathrm{tr}(\rho_1+\rho_2+\rho_3) = 3, \\
&\mathrm{tr}[(\rho_1+\rho_2+\rho_3)^2] = \frac{9}{2}, \\
&\mathrm{tr}[(\rho_1+\rho_2+\rho_3)^3] = \frac{15}{2} + \frac{3}{4}\cos\phi.
\end{aligned} \tag{36}$$

Since the eigenvalues of a 3×3 matrix are determined by its lowest three moments, the eigenvalues of M are determined by ϕ. Thus ϕ is the only independent invariant associated with the three vectors.

Table 3. Geometric phases $\phi = |\arg[\mathrm{tr}(\rho_1\rho_2\rho_3)]|$ associated with five different triples of vectors respectively of the SIC POVM generated from the fiducial vector in (29) for $t \in [0, \frac{\pi}{9}]$, where $\rho_j = |\psi_j\rangle\langle\psi_j|$ for $j = 1, 2, 3$, and $|\psi_j\rangle$s are three different vectors in the SIC POVM. Here $[Z]$ represents the vector $Z|\psi_{\mathrm{f}}(t)\rangle$, similarly for $[X]$ etc. Due to group covariance, $|\psi_{\mathrm{f}}(t)\rangle$ is chosen as $|\psi_1\rangle$. There are 28 different choices in total for the pair $|\psi_2\rangle, |\psi_3\rangle$. The second column shows the numbers of choices that lead to the specific geometric phases given in the third column.

$\{\lvert\psi_1\rangle, \lvert\psi_2\rangle, \lvert\psi_3\rangle\}$	multiplicity	geometric phase
$\{[I], [Z], [Z^2]\}$	1	$\phi_1 = \pi$
$\{[I], [X], [Z]\}$	18	$\phi_2 = \frac{\pi}{3}$
$\{[I], [X], [X^2]\}$	3	$\phi_3 = \pi - 3t$
$\{[I], [X], [X^2Z]\}$	3	$\phi_4 = \frac{\pi}{3} - 3t$
$\{[I], [X], [X^2Z^2]\}$	3	$\phi_5 = \frac{\pi}{3} + 3t$

Given a SIC POVM generated from the fiducial vector $|\psi_{\mathrm{f}}(t)\rangle$ in (29), due to group covariance, without loss of generality, we may assume $\rho_1 = \rho_f = |\psi_{\mathrm{f}}(t)\rangle\langle\psi_{\mathrm{f}}(t)|$. There are $\binom{8}{2} = 28$ different choices for the remaining two vectors, $|\psi_2\rangle, |\psi_3\rangle$. However, some analysis of the symmetry group of the SIC POVM reveals that ϕ may take at most five different values. Table 3 shows the five distinct geometric phases associated with five different triples of vectors in the SIC POVM on the orbit with $t \in [0, \frac{\pi}{9}]$. Figure 1 shows the variation of the five phases with t in a wider range. The two phases ϕ_1, ϕ_2 are independent of the parameter t. The other three phases ϕ_3, ϕ_4, ϕ_5 are periodic functions of t with the same shape and period $\frac{2\pi}{3}$, but are shifted from each other by $\pm\frac{2\pi}{9}$. If we do not distinguish the three phases ϕ_3, ϕ_4, ϕ_5, then the pattern displays a period of $\frac{2\pi}{9}$, with an additional mirror symmetry about $t = \frac{k\pi}{9}$ for $k = 0, \pm 1, \pm 2, \ldots$. It is clear from the figure that any two SIC POVMs on two different orbits respectively with $t \in [0, \frac{\pi}{9}]$ are not equivalent. By contrast, the equivalence of SIC POVMs on the three orbits corresponding to $t, \frac{2\pi}{9} - t, \frac{2\pi}{9} + t$ respectively is underpinned.

Let $\phi_{\min}$ be the minimum of the five phases listed in table 3; then $0 \le \phi_{\min} \le \frac{\pi}{3}$, and there is a one-to-one correspondence between $\phi_{\min}$ and t within the interval $[0, \frac{\pi}{9}]$, which reads

$$\phi_{\min} = \frac{\pi}{3} - 3t. \tag{37}$$

Thus $\phi_{\min}$ uniquely specifies the equivalence class of a group covariant SIC POVM in dimension three. Unlike the parameter t, the phases ϕ_js and $\phi_{\min}$ are intrinsic quantities of the SIC POVM, which are independent of the parametrization. They are especially useful when the SIC POVM is not constructed from a fiducial vector or the information about the symmetry group is missing, such as in the case of "hidden" SIC POVMs to be studied in section 4.2.

From the values of the geometric phases in table 3 and figure 1, it is straightforward to construct an alternative proof that, for each SIC POVM on the orbit with $t \in [0, \frac{\pi}{3}]$ and $t \neq \frac{\pi}{9}, \frac{2\pi}{9}$, there are no additional unitary or antiunitary symmetry operations

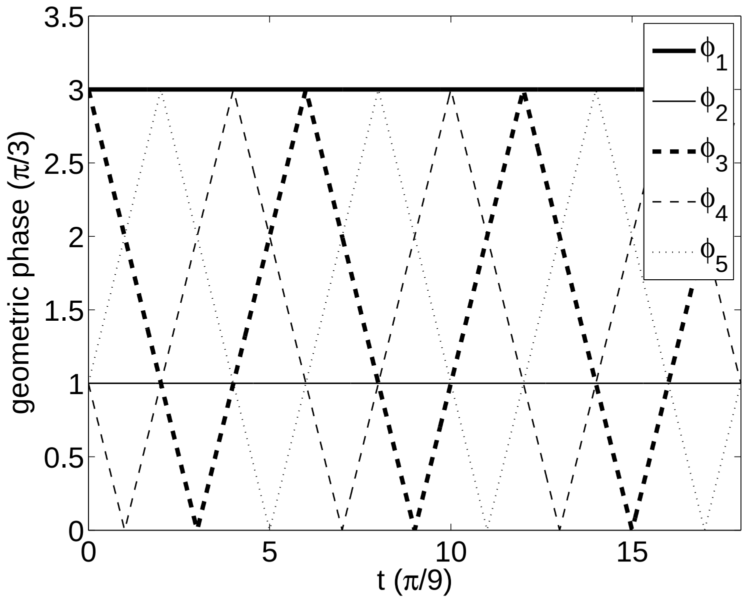

Figure 1. Geometric phases associated with five different triples of vectors respectively in the SIC POVM generated from the fiducial vector in (29) for $t \in [0, 2\pi]$. See the main text and table 3 for the meaning of the five different phases.

except those already contained in the standard extended Clifford group. The reasoning is based on the simple fact that these phases must be preserved under unitary or antiunitary operations. Interestingly, unlike the case in dimension two, it is not always possible to map three vectors to another three vectors in the same SIC POVM with unitary or antiunitary operations. In addition, in quantum state tomography with a SIC POVM in dimension three, the set of occurring probabilities is not permutation invariant, according to [16].

As far as tomographic efficiency is concerned, each SIC POVM is almost equally good [6, 7]. Although there are infinitely many inequivalent SIC POVMs in dimension three, and the set of occurring probabilities in tomography may depend on the SIC POVM chosen. It is interesting to know whether there is any other application, such that inequivalent SIC POVMs may lead to a drastic difference.

4.2. Uncover additional SIC POVMs by regrouping of fiducial vectors

Almost all known SIC POVMs are constructed from fiducial vectors under the action of the HW group. In this section, we uncover additional SIC POVMs in dimension three by regrouping of the fiducial vectors.

In addition to the SIC POVMs generated from the fiducial vectors in (29), there are some "hidden" SIC POVMs composed of vectors from different orbits or from different SIC POVMs on the same orbit. For example, the following nine vectors $X^j|\psi_{\mathrm{f}}(t_j)\rangle, ZX^j|\psi_{\mathrm{f}}(t_j)\rangle, Z^2X^j|\psi_{\mathrm{f}}(t_j)\rangle$ for $j = 0, 1, 2$ with $t_j \in [0, 2\pi)$ also form a SIC POVM. Although this SIC POVM is not constructed from a fiducial vector with

the HW group, it is equivalent to a SIC POVM on the orbit with $t = \frac{t_0+t_1+t_2}{3}$, under the unitary transformation $\mathrm{diag}(1, \mathrm{e}^{\mathrm{i}(t-t_2)}, \mathrm{e}^{\mathrm{i}(2t-t_0-t_2)})$.

Now suppose the eight SIC POVMs on each generic orbit are divided into four pairs as in section 4.1 (see (34)). In each pair of SIC POVMs, we can construct six additional SIC POVMs by regrouping of the 18 fiducial vectors. For example, given $k_1 = 0, 1, 2$, the following nine vectors

$$Z^{k_2}X^{k_1}|\psi_{\mathrm{f}}(t)\rangle, \qquad Z^{k_2}X^{k_1+1}|\psi_{\mathrm{f}}(t)\rangle, \qquad Z^{k_2}X^{k_1+2}|\psi_{\mathrm{f}}(-t)\rangle \qquad \text{for} \quad k_2 = 0, 1, 2 \tag{38}$$

in the first pair of SIC POVMs form a SIC POVM, which is unitarily equivalent to any SIC POVM on the orbit with $t' = \frac{t}{3}$. Similarly, the following nine vectors

$$Z^{k_2}X^{k_1}|\psi_{\mathrm{f}}(t)\rangle, \qquad Z^{k_2}X^{k_1+1}|\psi_{\mathrm{f}}(-t)\rangle, \qquad Z^{k_2}X^{k_1+2}|\psi_{\mathrm{f}}(-t)\rangle \qquad \text{for} \quad k_2 = 0, 1, 2 \tag{39}$$

also form a SIC POVM, which is unitarily equivalent to any SIC POVM on the orbit with $t'' = -\frac{t}{3}$, and is thus also unitarily equivalent to any SIC POVM on the orbit with $t' = \frac{t}{3}$. By the same token, six additional SIC POVMs can be obtained by regrouping of the 18 fiducial vectors from any other pair of SIC POVMs. Further analysis shows that these 24 additional SIC POVMs exhaust all SIC POVMs that can be obtained by regrouping of the 72 fiducial vectors on each generic orbit. All the 24 additional SIC POVMs are unitarily equivalent, however, they are not unitarily (even antiunitarily) equivalent to the original eight SIC POVMs. For the exceptional orbit with $t = \frac{\pi}{3}$, no SIC POVMs can be obtained by regrouping of the fiducial vectors in the four SIC POVMs.

Although the SIC POVMs on the three orbits with $t, \frac{2\pi}{9} - t, t + \frac{2\pi}{9}$ respectively for $t \in (0, \frac{\pi}{9})$ are unitarily equivalent (see section 4.1), the additional SIC POVMs obtained by regrouping of the fiducial vectors for the three orbits respectively are not unitarily (even antiunitarily) equivalent. This implies in particular that there is no unitary or antiunitary transformation that can map all SIC POVMs on one of the three orbits to that on another one.

SIC POVMs obtained by regrouping of fiducial vectors have been known for dimension four [13, 35]. For other dimensions, as far as the SIC POVMs found by Scott and Grassl [4] are concerned, such additional SIC POVMs can be obtained only for the orbits 8b and 12b (according to the labeling Scheme of Scott and Grassl) [35]. The peculiarity of SIC POVMs on these orbits is still a mystery.

5. Summary

The equivalence relation of SIC POVMs on different orbits of the (extended) Clifford group has been an elusive question in the community. So is the closely related question: Is the (extended) symmetry group of an HW covariant SIC POVM a subgroup of the (extended) Clifford group? In this paper we resolve these open questions for all prime dimensions. More specifically, we prove that, in any prime dimension not equal to three, each group covariant SIC POVM is covariant with respect to a unique HW group; its (extended) symmetry group is a subgroup of the (extended) Clifford group. Hence, SIC POVMs on different orbits are not equivalent. In dimension three, each group

covariant SIC POVM may be covariant with respect to three or nine HW groups; its symmetry group is a subgroup of at least one of the Clifford groups associated with these HW groups respectively. There may exist two or three orbits of equivalent SIC POVMs depending on the order of the symmetry group.

In addition, we establish a complete equivalence relation among group covariant SIC POVMs in dimension three, and classify inequivalent ones according to the geometric phases associated with fiducial vectors. Also, we uncover additional SIC POVMs by regrouping of the fiducial vectors from different SIC POVMs which may or may not be on the same orbit of the extended Clifford group. The picture of the SIC POVMs in dimension three is now more complete. The methods employed are also applicable to SIC POVMs in other dimensions.

Our results are an important step towards understanding the structure of SIC POVMs in prime dimensions. It would be highly desirable to extend these results to other dimensions.

Acknowledgements

The author is grateful to Markus Grassl for stimulating discussions and valuable comments and suggestions on the manuscript, and to Berthold-Georg Englert and Andrew Scott for stimulating discussions. The Centre for Quantum Technologies is funded by the Singapore Ministry of Education and the National Research Foundation as part of the Research Centres of Excellence programme.

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